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«On homogenization of attractors to the reaction-diffusion equations in a domain with rough boundary»

Abstract of the dissertation submitted for the degree of Doctor of Philosophy (PhD) in the educational program 8D05401 – "Mathematics"

Relevance of the research. Modern materials science and applied problems in physics, biology, and chemistry necessitate the study of processes occurring in micro inhomogeneous media (for example, in framework structures, porous media, composite materials with nanostructures, etc.). Solving such problems by numerical methods and computational tools is highly challenging, as they require the analysis and solution of algebraic systems with billions of unknowns. In such situations, methods of asymptotic analysis and homogenization theory become useful, as they allow one to derive simplified problems that are structurally much less complex yet remain close to the original problems in terms of their solutions.

Attractors describe the long-time behavior of solutions to dissipative nonlinear evolutionary equations and characterize the stability or instability of the limiting structures of the corresponding dynamical systems. In this work, the asymptotic behavior of attractors of a reaction–diffusion system with rapidly oscillating terms, both in the equation itself and in the boundary conditions, is studied in domains with locally periodic and randomly oscillating boundaries. The existence of trajectory attractors is established, and the limit (homogenized) reaction–diffusion systems are derived for both the locally periodic case and the statistically homogeneous randomly oscillating boundary case.

In this PhD thesis, the weak convergence ("almost surely" in the stochastic case, i.e., with probability one) of the trajectory attractors $\mathfrak{A}_\varepsilon$ of reaction–diffusion systems in domains with oscillating boundaries to the trajectory attractors $\overline{\mathfrak{A}}$ of the homogenized (limit) systems as $\varepsilon \to 0$ is established. Here, the small parameter ε characterizes the period and amplitude of the boundary oscillations. The parameter ε also appears, to a certain extent, in the third boundary condition posed on a part of the locally periodic oscillating boundary. Depending on the relationship between the powers of the small parameter in the boundary condition, three different limiting problems arise (the critical, subcritical, and supercritical cases). In the stochastic setting, the parameter ε also characterizes the microscopic heterogeneity of the domain boundary.

Objective of the study. The objective of this work is to investigate the asymptotic behavior of the trajectory attractors of the initial–boundary value problem for a reaction–diffusion system with rapidly oscillating terms in a domain with an oscillating boundary, as the small parameter governing the boundary oscillations tends to zero.

Methods of research. To address the posed problems, methods of asymptotic analysis and homogenization theory for initial–boundary value problems for partial differential equations are employed. In addition, tools from the qualitative theory of nonlinear partial differential equations and methods of functional analysis are used.

Scientific novelty. The following new scientific results have been obtained in this work:

1. The limit behavior of the trajectory attractors of a reaction–diffusion system is described depending on the relationship between the parameters β and $1-\alpha$, which represent the powers of the small parameter appearing with the unknown functions in the third boundary condition on the locally oscillating boundary.
2. The limit behavior of the trajectory attractor of the reaction–diffusion system with a third boundary condition on a randomly oscillating boundary is described.

Description of the main results.

In the second chapter of the PhD thesis, conditions are established for the convergence of the trajectory attractor of a reaction–diffusion system in a domain with a rapidly oscillating boundary to the trajectory attractor of the limit (homogenized) problem in a domain with a smooth (flat) boundary. It is shown that, depending on the nature of the oscillations and the parameters of the problem, as the small parameter tends to zero, the third boundary condition may converge to one of three different boundary conditions: the third boundary condition (Fourier condition) in the critical case, the second boundary condition (Neumann condition) in the subcritical case, or the first boundary condition (Dirichlet condition) in the supercritical case. The corresponding theorems on the convergence of trajectory attractors are formulated and proved.

In the third chapter, conditions for the convergence of the trajectory attractor of a reaction–diffusion system in a domain with a randomly oscillating boundary are obtained. A theorem is formulated proving the convergence of the trajectory attractors of the original initial–boundary value problem to the trajectory attractor of the corresponding homogenized (limit) problem.

Justification of the novelty and significance of the obtained results.

The scientific results obtained in this work are new and have pure theoretical character. They describe the long-term (time-asymptotic) behavior of solutions to a reaction–diffusion system in domains with locally periodic and randomly oscillating boundaries. The results are also of interest from the standpoint of applied mathematics and may be used in the numerical modeling of transport processes, such as the motion of fluids or gases in media with rapidly oscillating (rough) boundaries.

In addition, the scientific results of this work may be used in the educational process for the training of researchers in master's and doctoral programs within elective courses on partial differential equations, homogenization theory, and the qualitative theory of nonlinear equations.

New scientific results obtained in the work.

The following new scientific results have been obtained in this PhD thesis:

1. The limit behavior of the trajectory attractors of a reaction–diffusion system is described depending on the relationship between the parameters β and $1-\alpha$, which represent the powers of the small parameter appearing with the unknown functions in the third boundary condition on the locally oscillating boundary.

2. The limit behavior of the trajectory attractor of the reaction–diffusion system with a third boundary condition on a randomly oscillating boundary is described.

Publications.

The main results of the PhD thesis are published in 11 works (4 papers and 7 conference abstracts). Among them, 4 papers are published in journals indexed in the Web of Science and Scopus databases (3 papers in journals with a percentile above 35, and 2 articles in international journals).

1. Homogenization of attractors to reaction-diffusion equations in domains with rapidly oscillating boundary: Critical case // Networks and Heterogeneous Media. – 2024. – Vol. 19, Iss. 3. – P. 1381–1401 (IF2024=1.3, Q3; CiteScore2024=1.9, percentile 43).

2. Homogenization of attractors to reaction-diffusion equations in domains with rapidly oscillating boundary: Supercritical case // Ufa Mathematical Journal. – 2025. – Vol. 17, Iss. 2. – P. 91–104 (IF2024=0.4, Q4; CiteScore2024=1.2, percentile 43).

3. Homogenization of attractors to reaction-diffusion equations in domains with rapidly oscillating boundary: Subcritical case // Bulletin of the Karaganda University. Mathematics series. – 2025. – Vol. 118, Iss. 2. – P. 28–43 (IF2024=0.9, Q2; CiteScore2024=1.4, percentile 53).

4. Homogenization of attractors to the reaction-diffusion system in a domain with rough boundary // Journal of Mathematics, Mechanics and Computer Science. – 2025. – Vol. 126, Iss. 2. – P. 3–24 (IF2024=0.3, Q4; CiteScore2025=0.4 percentile 18).

Abstracts in the proceedings of international conferences.

1. Об усреднении аттракторов системы уравнений реакции-диффузии в области с шероховатой границей // XIX International Scientific Conference of Students, Master's Students, and Young Researchers "Lomonosov – 2024", Kazakhstan Branch of Lomonosov Moscow State University (Astana, 2024, pp. 12–13).

2. Об усреднении аттракторов системы уравнений реакции-диффузии в области с шероховатой границей // Proceedings of the International Scientific and Practical Conference dedicated to the 270th anniversary of Moscow University, Kazakhstan Branch of Lomonosov Moscow State University (Astana, 2024, pp. 41–48).

3. Об усреднении аттракторов системы уравнений реакции-диффузии в области с шероховатой границей // Current Problems of Analysis, Differential Equations, and Algebra (EMJ-2025). Proceedings of the International Conference

dedicated to the 15th anniversary of the journal "Eurasian Mathematical Journal", L.N. Gumilyov Eurasian National University (Astana, 2025, pp. 76–77).

4. Об усреднении аттракторов системы уравнений реакции-диффузии в области со случайной осциллирующей границей // XX International Scientific Conference of Students, Master's Students, and Young Researchers "Lomonosov – 2025", Kazakhstan Branch of Lomonosov Moscow State University (Astana, 2025, pp. 12–15).

5. Об усреднении аттракторов системы уравнений реакции-диффузии в области с шероховатой границей // Republican Scientific Conference dedicated to the 80th anniversary of the birth of Academician Sh.A. Alimov, Mirzo Ulugbek National University of Uzbekistan (Tashkent, 2025, pp. 179–180).

6. Homogenization of trajectory attractors of random reaction-diffusion systems in domains with rapidly oscillating boundary // International Conference dedicated to the distinguished mathematician I.G. Petrovsky, Moscow State University (Moscow, 2025, pp. 12–13).

7. Homogenization of trajectory attractors of random reaction-diffusion systems in domains with rapidly oscillating boundary // 15th ISAAC Congress, Nazarbayev university, (Astana, 2025. – P. 167).

Approbation of the obtained results.

The main results of the dissertation were presented at the following conferences:

1. Об усреднении аттракторов системы уравнений реакции-диффузии в области с шероховатой границей // International Scientific Conference "Mathematics in the Constellation of Sciences," dedicated to the 85th anniversary of Academician of the Russian Academy of Sciences V.A. Sadovnichy, Kazakhstan Branch of Lomonosov Moscow State University (Astana, April 1–2, 2024).

2. Об усреднении аттракторов системы уравнений реакции-диффузии в области с шероховатой границей // XIX International Scientific Conference of Students, Master's Students, and Young Researchers "Lomonosov – 2024," dedicated to the 270th anniversary of Moscow University, Kazakhstan Branch of Lomonosov Moscow State University (Astana, April 19–20, 2024).

3. Об усреднении аттракторов системы уравнений реакции-диффузии в области с шероховатой границей // International Scientific and Practical Conference "Fundamental Science and the Priorities of the 21st Century," dedicated to the 270th anniversary of Moscow University. Kazakhstan Branch of Lomonosov Moscow State University (Astana, November 29, 2024).

4. Об усреднении аттракторов системы уравнений реакции-диффузии в области с шероховатой границей // "Current Problems of Analysis, Differential Equations, and Algebra" (EMJ-2025). International Conference dedicated to the 15th anniversary of the journal "Eurasian Mathematical Journal", L.N. Gumilyov Eurasian National University (Astana, January 7–11, 2025).

5. Об усреднении аттракторов системы уравнений реакции-диффузии в области со случайной осциллирующей границей // XX International Scientific

Conference of Students, Master’s Students, and Young Researchers “Lomonosov – 2025”, Kazakhstan Branch of Lomonosov Moscow State University (Astana, April 11–12, 2025).

6. Об усреднении аттракторов системы уравнений реакции-диффузии в области с шероховатой границей // Republican Scientific Conference “Modern Methods of Mathematical Physics and Their Applications,” dedicated to the 80th anniversary of Academician Sh.A. Alimov, Mirzo Ulugbek National University of Uzbekistan (Tashkent, April 22–24, 2025).

7. Homogenization of trajectory attractors of random reaction-diffusion systems in domains with rapidly oscillating boundary // International Conference “Differential Equations and Related Topics,” dedicated to the distinguished mathematician I.G. Petrovsky, Lomonosov Moscow State University (Moscow, May 19–23, 2025).

8. Homogenization of trajectory attractors of random reaction-diffusion systems in domains with rapidly oscillating boundary // 15th ISAAC Congress (Astana: Nazarbayev university, 21-25 июля 2025 года)

In addition, the results of the work were discussed at the following scientific seminars:

1. Scientific Seminar on Equations of Mathematical Physics, Faculty of Mechanics and Mathematics, Lomonosov Moscow State University. Supervisor: G.A. Chechkin (Moscow, September 18, 2024).

2. Scientific Seminar “Functional Analysis and Its Applications.” Supervisors: Academician of the National Academy of Sciences of the Republic of Kazakhstan M. Otelbaev, Academician of the National Academy of Sciences of the Republic of Kazakhstan R.O. Oinarov, Professors E.D. Nursultanov and K.N. Ospanov (Astana, March 13, 2025; June 5, 2025).

3. Scientific Seminar of the Department of “Fundamental Mathematics,” L.N. Gumilyov Eurasian National University (Astana, September 25, 2025).

Contribution of the doctoral student to the preparation of each publication. The main results of the dissertation were published in four scientific papers. All papers were prepared in co-authorship. In these works, the formulation of the problems and the choice of research methodology were proposed by the scientific advisors, while the doctoral student independently formulated the main and auxiliary results and provided their proofs.

Structure and volume of the dissertation.

The PhD thesis consists of an introduction, a main part comprising three chapters, a conclusion, and a list of references. The numbering of formulas is arranged as follows: the first digit denotes the chapter number, the second denotes the subsection number, and the third denotes the sequential number of the formula within that subsection. The numbering of theorems, lemmas, propositions, and remarks is also three-level and follows the same scheme. The total length of the dissertation is 100 pages.

Main content of the work.

The introduction provides a justification of the relevance and novelty of the topic and formulates the aims, object, subject, and research tasks. It also includes a list of publications related to the dissertation, as well as a list of conferences and seminars where the research results were presented.

The first chapter of the PhD thesis presents the basic concepts related to trajectory attractors of autonomous evolutionary equations. Specifically, it includes the definition of a trajectory attractor, the scheme for its construction, and the corresponding statements. This chapter also introduces the probabilistic structure describing the random oscillation of the boundary. Furthermore, it formulates and justifies the conditions under which Birkhoff's ergodic theorem can be applied.

The second chapter considers the homogenization problem in a micro-heterogeneous domain with a rapidly oscillating boundary. It is assumed that a system of nonlinear reaction–diffusion equations with rapidly oscillating terms and dissipation is posed in the domain. On the locally periodic oscillating part of the boundary, a third boundary condition with rapidly oscillating coefficients depending on a small parameter characterizing the boundary oscillation is imposed. Depending on the relationship between the powers of the small parameter in the boundary condition, different homogenized (limiting) problems arise (the critical, subcritical, and supercritical cases). For each case, a theorem on the convergence of the trajectory attractors of the original reaction–diffusion system to the trajectory attractors of the homogenized (limit) system is proved. Moreover, in the case of uniqueness of the solution to the system, the convergence of global attractors is also established.

The third chapter examines a reaction–diffusion system in a domain with a randomly rapidly oscillating boundary. In the case of a statistically homogeneous random boundary structure, a homogenized reaction–diffusion system with deterministic (non-random) coefficients is obtained. As in the previous chapter, a theorem is proved establishing the convergence of the trajectory and global attractors of the original Cauchy problem to the trajectory and global attractors of the corresponding limit problem.